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ON PACKING BIPARTITE GRAPHS

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G and H, two simple graphs, can be packed if G is isomorphic to a subgraph of $\overline{H}$, the complement of H. A theorem of Catlin, Spencer and Sauer gives a sufficient condition for the existence of packing in terms of the product of the maximal degrees of G and H. We improve this theorem for bipartite graphs. Our condition involves products of a maximum degree with an average degree. Our relaxed condition still guarantees a packing of the two bipartite graphs.

0. Introduction

If G is a graph, then $V(G)$, $E(G)$, $D(G)$, $\delta(G)$, $\overline{d}(G)$ will denote its vertex set, edge set, maximal degree, minimal degree and average degree. Let U be any subset of $V(G)$. Let $D_U(G)$ and $\overline{d}_U(G)$ be the maximum and average degrees where the maximum and average taken over the vertices in U (the corresponding degrees are based on the whole graph). Let G_v be the collection of graphs with vertex set of size v. Let $B_{u,w}$ be the collection of bipartite graphs with two color classes of size u and w. If f be a 1-1 map from $V(G)$ onto W, let G^f be the image of G under f, i.e. a graph on the vertex set W.

Definition 0.1 (a) Let $G,H \in G_v$. A *packing* is a bijection $f: V(H) \to V(G)$ such that the edge set of G and H^f are disjoint.

(b) Let $G,H \in B_{u,w}$. Let us assume that G has color classes U and W and H has color classes U' and W'. A *bipartite packing* is a bijection f that maps U' to U and W' to W such that the edge set of G and H^f are disjoint.

Packing graphs is a heavily studied subject in graph theory. A good survey of this research can be found in [2]. Next we summarize the known results on packing.

Much effort has been spent for packing sparse graphs [15], [17], [5], [6], [13], [16], [10]. A typical theorem from this area is:

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Theorem 0.2. (N. Sauer and J. Spencer [15]) *If $|E(G)|$, $|E(H)| \le v-2$ (where $|V(G)| = |V(H)| = v$) then G and H can be packed.*

One can extend packing to packing several graphs. We just refer the reader to [11], and we mention a nice conjecture from this paper.

Conjecture 0.3. (A. Gyárfás and J. Lehel [11]) *Let T_k be any tree with vertex set of size k $(k=1,\dots,n)$. Then there is a packing of $T_1, T_2, \dots, T_n$ into the complete graph on n vertices.*

The following few theorems give sufficient conditions on the number of edges for the existence of a packing.

Theorem 0.4. (B. Bollobás and S.E. Eldridge [4]) *If $|E(G)| + |E(H)| \le \lfloor \frac{3}{2}(v-1) \rfloor$ (where $|V(G)| = |V(H)| = v$) then there is a packing of G and H.*

For improvements (but still with a linear upper bound in the condition on the sum of the number of edges) see [4].

Theorem 0.5. (B. Bollobás and S.E. Eldridge [4])

(i) *If $G, H \in G_v$ and $|E(G)||E(H)| < \binom{v}{2}$ then G and H can be packed.*

(ii) *If $G, H \in B_{u,w}$ and $|E(G)||E(H)| < uw$, then G and H can be packed as bipartite graphs.*

Theorem 0.6. (B. Bollobás and S.E. Eldridge [4])

(i) *If $G, H \in G_v$, $|E(H)| < \frac{v}{3}$ and $|E(G)| < \frac{1}{15} v^{\frac{3}{2}}$ then G and H can be packed.*

(ii) *If $G, H \in B_{u,u}$, $|E(H)| < \frac{u}{3}$ and $|E(G)| < \frac{1}{15} u^{\frac{3}{2}}$ then G and H can be packed as bipartite graphs.*

For us the most important sufficient conditions will be the following ones on the maximal degrees.

Theorem 0.7. (Conditions on the maximal degree [15],[7])

(i) *If G, $H \in G_v$ and $D(G)D(H) < v/2$ then G and H can be packed.*

(ii) *If $A, B \in B_{u,w}$ and $D_U(A)D_W(B) + D_W(A)D_U(B) < u$, then A and B can be packed as bipartite graphs.*

In the last two statements the bounds in the conditions are tight (up to negligible factors). For Theorem 0.7.(ii) (the bipartite case) this can be easily shown using the probabilistic method in [15]. For Theorem 0.7.(i) there is an easy construction [4] showing that one cannot improve the condition with more than a factor of 2. That example suggests the following conjecture.

Conjecture 0.8. (B. Bollobás and S.E. Eldridge [4]) *Let G and H be two graph on a vertex set of size v. If $(D(G)+1)(D(H)+1) \le v+1$ then there is a packing of G and H.*

The condition in Theorem 0.7.(ii) restricts the product of the maximum degrees of G and H. Our improvement comes from relaxing one of the terms to average degree.

Theorem 0.9. *Let $G, H \in G_{u,w}$. Assume that*

(a) $u \le w \le 2u$,
(b) $\bar{d}_{U'}(G) D_W(H) \le u/100$,
(c) $\bar{d}_U(H) D_{W'}(G) \le u/100$,
(d) $D_U(G), D_{U'}(H) \le u/1000 \log u$.

Then G and H can be packed.

This research was motivated by questions on decision tree complexity. For an application of this theorem in this direction see [12].

1. The improved packing theorem for bipartite graphs

In this section we prove the result stated in the introduction.

First, we review Catlin's idea. Given $G, H \in G_{u,w}$ with color classes U, W and U', W', resp. We want to find a sufficient condition for existence a packing. We take an arbitrary bijection $f: U' \to U$. Define a bipartite graph between W and W' based on whether two nodes can be identified or not. Now the problem is simply finding a matching in this auxiliary graph.

Definition 1.1 Let $G, H \in G_{u,w}$. Let U, W, U' and W' be the corresponding color classes. Given f, a bijection $U' \to U$, we define a bipartite graph B_f with color classes W and W'. We make $x \in W$ and $y \in W'$ adjacent iff x and y can be identified, i.e., the neighborhoods of x in G and of y in H^f are disjoint subsets of U.

Now it is easy to show that if G and H satisfy the condition of Theorem 0.7.(ii) then for any bijection f B_f satisfies the condition of König's theorem (see e.g. [14], Chap. 7, prob. 4.) and therefore possesses a perfect matching along which we can map W' to W to obtain a packing. It is worth to state this fact as a separate lemma.

Lemma 1.2. (i) *Let $G \in B_{u,u}$. If $\delta_U(G), \delta_W(G) \ge u/2$ then G has a perfect matching.*
(ii) *Let $G \in B_{u,u}$. If $\delta_U(G) + \delta_W(G) \ge u$ then G has a perfect matching.*

The proof of our result is probabilistic. Our goal is to show that there exists a bijection $f: U \to U'$ such that B_f has a perfect matching. We are going to show that this is true for a random bijection.

Let $d_1, \dots, d_u$ be all the degrees in U, and let $e_1, \dots, e_u$ be all the degrees in U'.

Lemma 1.3. *Let $f: U \to U'$ be a random bijection, all bijections being equally likely.*

$$\textit{Prob}(B_f \text{ has perfect matching}) \ge 1 - w\,\textit{Prob}\left(\sum_{i \in R} d_i \ge \frac{w}{2}\right) - w\,\textit{Prob}\left(\sum_{i \in S} e_i \ge \frac{w}{2}\right),$$

where S is a random subset of U' of size $D_W(G)$, all such subsets being equally likely, and R is a random subset of U of size $D_{W'}(G)$, all such subsets being equally likely.

Proof. We are interested in the event

$$E = B_f \text{ has a perfect matching}.$$

By Lemma 1.2 the following event is a subset of E.

$$F = \text{ Each node of } B_f \text{ has degree at least } \frac{w}{2}.$$

One elementary bad event is

$$F_x = x \text{ has degree in } B_f \text{ less than } \frac{w}{2} \text{ (for } x \in W \cup W').$$

Using this notation

$$E \supseteq F = \Omega - \cup_{x \in W \cup W'} F_x.$$

Thus

$$Prob(E) \geq 1 - \sum_{x \in W \cup W'} Prob(F_x).$$

For $x \in W$, F_x is exactly the event that the image $f(N(x))$ of $N(x)$ ($f(N(x)) \subset U'$) has a neighborhood in W' of size more than $\frac{w}{2}$. The event that the sum of the degrees in $f(N(x))$ is at least $\frac{w}{2}$ is a superset of F_x. If $x \in W$ then $f(N(x))$ is a random set of size $|N(x)|$ and its size is at most $D_W(G)$. This completes the proof. ∎

Our conditions on G and H are symmetric. So it is enough to show that

$$Prob\left(\sum_{i \in R} d_i \geq \frac{w}{2}\right) < \frac{1}{2w}.$$

R is a random subset of U'. There are different models for random sets. In our case R is a random set of a given size. Another model is that each element of our universe will be in the set with a given probability. This model is more convenient. It is well-known in the theory of random graphs [3] that by choosing the right parameters the two models yield basically the same theorems. So our next step is to change to the second model. For this we need some inequality for Bernoulli random variables.

Lemma 1.4. (Chernoff [8]) *Let $X_1, X_2, \dots, X_N$ be independent* 0–1 *random variables such that $Prob(X_i = 1) = p$. If $m \geq Np$ is an integer then*

$$Prob\left(\sum_{i=1}^{N} X_i \geq m\right) \leq \left(\frac{Np}{m}\right)^m \exp(m - Np).$$

An easy consequence of this is the following.

Lemma 1.5. ([9], [1]) *Let $X_1, X_2, \dots, X_N$ be independent* 0–1 *random variables such that $Prob(X_i = 1) = p$. Then for every $0 < \beta < 1$,*

(i) $Prob\left(\sum_{i=1}^{N} X_i \leq \lfloor (1-\beta) Np \rfloor\right) \leq \exp\left(-\frac{\beta^2 Np}{2}\right)$.

(ii) $Prob\left(\sum_{i=1}^{N} X_i \leq \lfloor (1+\beta) Np \rfloor\right) \leq \exp\left(-\frac{\beta^2 Np}{3}\right)$.

And now let us see the reduction.

Lemma 1.6. *Let $X_1,\dots,X_u$ be independent random variables such that*

$$Prob(X_i = d_i) = p > 2\frac{D_W(G)}{u} \quad \text{and} \quad Prob(X_i = 0) = 1 - p.$$

Let Δ be a random subset of $1,2,\dots,u$ of size $D_W(G)$. Then

$$Prob\left(\sum_{i\in\Delta} d_i \geq \frac{w}{2}\right) < 2\ Prob\left(\sum_{i=1}^{u} X_i \geq \frac{w}{2}\right).$$

Proof. Let Δ_i be a random subset of $1,2,\dots,u$ of size i, with all i-subsets of $1,2,\dots,u$ being equally likely. Let $P_i = Prob\left(\sum_{j\in\Delta_i} d_j > \frac{w}{2}\right)$. Then $P_0 \leq P_1 \leq \dots \leq P_u$.

Then

$$\begin{aligned}
Prob\left(\sum_{i=0}^{u} X_i > \frac{w}{2}\right) &= \sum_{k=1}^{u} \binom{u}{k} p^k (1-p)^{u-k} P_k \\
&\geq P_{\lfloor \frac{1}{2}up \rfloor} \sum_{\lfloor \frac{1}{2}np \rfloor \leq k \leq \lfloor \frac{3}{2}up \rfloor} \binom{u}{k} p^k (1-p)^{u-k} \\
&\geq \frac{1}{2} P_{\lfloor \frac{1}{2}up \rfloor} \geq \frac{1}{2} P_{D_W(G)} \\
&= \frac{1}{2} Prob\left(\sum_{i\in\Delta} d_i > \frac{w}{2}\right).
\end{aligned}$$

∎

So at this point using the notation of the previous lemma, we will give an upper bound on $Prob\left(\sum_{i=1}^{u} X_i \geq \frac{w}{2}\right)$.

Let us fix the value of p to be $10\frac{D_W(G)}{u}$. Notice that the conditions of Theorem 0.9 imply

$$\frac{w}{2} \gg E\left(\sum_{i=1}^{u} X_i\right) = \sum_i 10\frac{D_W(G)}{u} d_i = 10 D_W(G) \bar{d}_{U'}.$$

So we need an upper bound on the probability that a sum of independent random variables is much greater than their expected sum. The Chernoff bound is that kind of result, but it is about Bernoulli random variables. We use the method of the proof of Chernoff's theorem to get the desired upper bound. For that we need the notion of characteristic function.

Definition 1.7 Let X be a random variable. Its characteristic function is e^{tX}, a random variable depending on the real parameter t.

The following lemma shows an important property of the characteristic function.

Lemma 1.8. *Let $X_1,\dots,X_N$ be independent random variables. Then*

$$E\left(\prod_{i=1}^{N} e^{tX_i}\right) = \prod_{i=1}^{N} E\left(e^{tX_i}\right).$$

Now we have everything required to prove the last lemma that we need.

Lemma 1.9. *Let $0 \le d_1, d_2, \ldots, d_u \le L = \frac{u}{1000 \log u}$ be integers and define $\bar{d}$ by $\sum_{i=1}^{u} d_i = \bar{d}u$. Let $X_1, X_2, \ldots, X_u$ be independent random variables such that $Prob(X_i = d_i) = p$ and $Prob(X_i = 0) = 1 - p$. Then $Prob\left(\sum_{i=1}^{u} X_i > 10p\bar{d}u\right) \le \frac{1}{u^2}$.*

Proof. For all positive t

$$Prob\left(\sum_i X_i > 10p\bar{d}u\right) = Prob\left(e^{(\sum_i X_i)t} > e^{10p\bar{d}ut}\right).$$

Let us compute $E(e^{\sum_i X_i t})$.

$$E(e^{\sum_i X_i t}) = E\left(\prod_i e^{X_i t}\right) = \prod_i E\left(e^{X_i t}\right)$$
$$= \prod_i \left(1 - p + pe^{d_i t}\right) = \prod_i (1 - p(1 - e^{d_i t})).$$

An easy calculation shows that this product is maximal if all d_i's are 0 or L, the maximal possible value of them. So

$$E(e^{\sum_i X_i t}) \le (1 - p(1 - e^{Lt}))^{\frac{\bar{d}u}{L}} < (1 - p(1 - (1 + 2Lt)))^{\frac{\bar{d}u}{L}} < (1 + 2pLt)^{\frac{\bar{d}u}{L}} < e^{2p\bar{d}ut},$$

assuming that $Lt \le 1$.

Using Markov's inequality

$$Prob\left(\sum_i X_i > 10p\bar{d}u\right) = Prob\left(e^{\sum_i X_i t} > e^{10p\bar{d}ut}\right) \le \frac{e^{2p\bar{d}ut}}{e^{10p\bar{d}ut}} = e^{-8p\bar{d}ut}.$$

Fixing the value of t to be $1/L$ our bounds are still true and we obtain the desired upper bound. ∎

We obtain the promised packing theorem (Theorem 0.9) as a corollary.

Proof of Theorem 0.9. Applying Lemma 1.9, Lemma 1.6 and Lemma 1.3 we obtain that for a random f that B_f has a perfect matching with positive probability . This proves that there exists a concrete bijection f such that the corresponding B_f has a perfect matching. This perfect matching is an identification of W and W', which together with f gives us a packing. ∎

Our proof heavily uses the fact that we are working with bipartite graphs. It is an interesting open question whether one can extend our result to the case of general graphs.

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